Close Wed: HW_5A, 5B (7.1, 7.2)
Close Fri: HW_5C (7.3)
Office Hours: 1:30-3:30 in THO 335
See posting of several more integration by parts examples (with solutions).

7.1 Integration by Parts (continued)

Summary: $\int u\,dv = uv - \int v\,du$

1. Pick u = ??. The rest is dv.
2. Differentiate to get du
   Integrate to get v.

Here are all the most common examples:

a) Products: xe^x, $x^2\cos(3x)$, $x\sin(5x)$
b) Logarithms: $\ln(x)$, $x^{10}\ln(x)$, …
c) Inv. Trig: $\sin^{-1}(x)$, $\tan^{-1}(x)$, …
d) Products: $e^x\sin(x)$, $e^x\cos(x)$

Entry Task: Evaluate

$$\int_0^1 \sin^{-1}(x)dx = \int_0^1 \arcsin(x)dx$$

deriv. $u = \sin^{-1}(x)$ → $du = \frac{1}{\sqrt{1-x^2}}dx$

integrate $dv = dx$ → $v = x$

$$= x\sin^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$

substitution!

$$= (1\sin^{-1}(1) - 0\sin^{-1}(0)) - \int_1^0 \frac{x}{\sqrt{u}}\frac{1}{-2x}du$$

$u = 1 - x^2$
$du = -2x\,dx$
$-\frac{1}{2x}du = dx$

$$= \frac{\pi}{2} + \frac{1}{2}\int_1^0 u^{-1/2}du$$

$$= \frac{\pi}{2} + \frac{1}{2}\left(2u^{1/2}\Big|_1^0\right)$$

$$= \frac{\pi}{2} + (0 - 1)$$

$$= \boxed{\frac{\pi}{2} - 1}$$

Example: (Never ending integration by parts and how to end it):

$$\int e^x \cos(x)\, dx$$

$u = e^x \qquad dv = \cos(x)\,dx$

$du = e^x dx \qquad v = \sin(x)$

$$= e^x \sin(x) - \int e^x \sin(x)\,dx$$

$u = e^x \qquad dv = \sin(x)\,dx$

$du = e^x dx \qquad v = -\cos(x)$

$$= e^x \sin(x) - \left(-e^x \cos(x) - \int -e^x \cos(x)\,dx\right)$$

Thus,

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

SAME UP TO A CONSTANT

THEREFORE

$$2 \int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + D$$

$$\int e^x \cos(x)\,dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + \frac{D}{2}$$

$$= \boxed{\frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C}$$

7.2 Trigonometric Integral Methods

Goal: Build up rules for integrating combinations of trig functions.

Basic motivating examples:
All of these could be done with substitution: What is *u*?

$$\int \sin^4(x)\cos(x)\,dx = \int u^4\,du \quad (\text{with } u = \sin(x))$$

$$\int \sin(x)\cos^3(x)\,dx = -\int u^3\,du \quad (\text{with } u = \cos(x))$$

$$\int \tan^5(x)\sec^2(x)\,dx = \int u^5\,du \quad (\text{with } u = \tan(x))$$

$$\int \sec^6(x)\sec(x)\tan(x)\,dx = \int u^6\,du \quad (\text{with } u = \sec(x))$$

Idea: Use trig identities to turn a problem into a substitution problem like those above.

Tools

Essential Trig Identities:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \cot(x) = \frac{\cos(x)}{\sin(x)},$$
$$\sec(x) = \frac{1}{\cos(x)}, \csc(x) = \frac{1}{\sin(x)}.$$

$$\sin^2(x) + \cos^2(x) = 1$$
$$\tan^2(x) + 1 = \sec^2(x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$
$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$
$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

See my online posting (or Appendix D of your book) for a more general discussion of trig identities.

Add these to your table of integrals you know (an updated table is already on the website):

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x)\,dx =$$
$$\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C$$

Examples

1

ODD POWER ON COSINE

$$\int \sin^2(x)\cos^3(x)\,dx$$

PULL OUT ONE COSINE

$$= \int \sin^2(x)\cos^2(x)\cos(x)\,dx$$

$$= \int \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx$$

USE identity

$u = \sin(x)$

$du = \cos(x)\,dx$

$$= \int u^2(1-u^2)\,du$$

$$= \int u^2 - u^4\,du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C}$$

2

ODD POWER ON SINE

$$\int \sin^3(x)\,dx$$

PULL OUT ONE SINE

$$= \int \sin^2(x)\sin(x)\,dx$$

$$= \int \left(1-\cos^2(x)\right)\sin(x)\,dx$$

USE identity

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -\int 1-u^2\,du$$

$$= -\left(u - \frac{1}{3}u^3\right) + C$$

$$= \boxed{-\cos(x) + \frac{1}{3}\cos^3(x) + C}$$

3 $\int \cos^2(x)\,dx$ EVEN POWER ON BOTH

$= \int \frac{1}{2}(1+\cos(2x))\,dx$ HALF ANGLE IDENTITY

$= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C$

$= \boxed{\frac{1}{2}x + \frac{1}{4}\sin(2x) + C}$

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4 $\int \sin^4(x)\,dx$ EVEN POWER ON BOTH

$= \int \sin^2(x)\sin^2(x)\,dx$ HALF ANGLE

$= \int \frac{1}{2}(1-\cos(2x))\,\frac{1}{2}(1-\cos(2x))\,dx$

$= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\,dx$ HALF ANGLE AGAIN!

$= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\,dx$

$= \frac{1}{4}\int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx$

$= \boxed{\frac{1}{4}\left(\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right) + C}$

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5 $\int \tan^2(x)\sec^4(x)\,dx$ EVEN POWER ON SECANT

$= \int \tan^2(x)\sec^2(x)\,\sec^2(x)\,dx$ PULL OUT $\sec^2(x)$

$= \int \tan^2(x)(\tan^2(x)+1)\sec^2(x)\,dx$ use identity

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$= \int u^2(u^2+1)\,du$

$= \int u^4 + u^2\,du$

$= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C}$

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6 $\int \tan^3(x)\sec^5(x)\,dx$ ODD POWER ON SEC(X)TAN(X)

$= \int \tan^2(x)\sec^4(x)\,\sec(x)\tan(x)\,dx$ PULL OUT $\sec(x)\tan(x)$

$= \int (\sec^2(x)-1)\sec^4(x)\sec(x)\tan(x)\,dx$ use identity

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$= \int (u^2-1)u^4\,du$

$= \int u^6 - u^4\,du$

$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$

$= \boxed{\frac{1}{7}\sec^7(x) - \frac{1}{5}\sec^5(x) + C}$